# Proof of Hopf-Thurston Conj: char $k = p > 0$ case.

**Thm 1.3** $X$ smooth proj $/\mathbb{C}$

Assume $\exists$ large rep $\rho : \pi_1(X) \to GL_n(k)$

for some field $k$ $\boxed{\text{of positive char } p}$

**Then** $\chi(X, P) \geq 0$, $\forall P \in \mathrm{Perv}(\underline{\mathbb{C}})$

Goal of today

(**Def** $\rho : \pi_1(X) \to GL_n(k)$ is called <u>large</u>, if

$\forall$ closed irr subvar $Z \subset X$,

$\# \mathrm{Im}(\pi_1(Z^{norm}) \to \pi_1(X) \xrightarrow{\rho} GL_n(k)) = \infty$ )

By global index formula

$$\chi(X, P) = CC(P) \cdot T^*_X X$$

where $CC(P) = \sum_i^{fin} n_i T^*_{Z_i} X$ is the characteristic cycle.

$\rightsquigarrow$ Proof of 1.3 reduces to show

Thm 1.5 $\quad T^*_Z X \cdot T^*_X X \geq 0$ in $CH_0(T^*X)$

$\forall Z \subset X$ closed irr subvar.

Here $T^*_Z X := \overline{T^*_{Z_{reg}} X}^{T^*X}$ conormal var

Proof of Thm 1.5

$\rho : \pi_1(X) \to GL_n(K)$ large

TODO $\implies$ Shafarevich morphism

$sh_\rho : X \longrightarrow Sh_\rho(X)$

is the same as $id_X$.

• [DY, Cor 2.10] $\Rightarrow sh_\rho = s_\sigma$ for some reductive rep $\sigma : \pi_1(X) \to GL_n(K')$

Katzarkov-Eyssidieux reduction map

for some $K'$ of char $p$.

$s_\sigma = id_X$

Last talk $\implies$ [DW Prop 2.10] spectral coverings

$\exists$ closed multivalued one form $\eta_\sigma$ s.t. $\forall Z \subset X$, irr closed subvar

$\eta_\sigma \big|_{Z_{reg}}$ is nontrivial.

So in particular, $\eta_\sigma$ is nontrivial when restricted to the regular locus of any closed subvar to come, e.g. the $Z_i$'s.

$\implies$ [DW Cor 3.10]

$$T_Z^*X \cdot T_X^*X = \sum^{\text{finite}} \lambda_i T_{Z_i}^*X \cdot T_X^*X$$

for some $\lambda_i \in \mathbb{Q}_{>0}$, $Z_i \subsetneq Z$ closed.

$\Longrightarrow$ Can assume all $Z_i$ are pts.

Induction on $\dim Z_i$

$\Longrightarrow$ $T_Z^* X \cdot T_X^* X > 0$ $\square$

$T_{Z_i}^* X \cdot T_X^* X = 1$

# Character variety : recollection.

$\Gamma$ = f.g. group

$R(\Gamma, n)$ = affine $\mathbb{Z}$-scheme of f.type representing n-dim'l rep of $\Gamma$ " of char 0 " (see next page)

i.e. $R(\Gamma, n)(A) = \mathrm{Hom}_{\mathrm{Grp}}(\Gamma, GL_n(A))$

Define $R(\Gamma, n)_{\mathbb{F}_p} := R(\Gamma, n) \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{F}_p$.

$\rightsquigarrow GL_n(\mathbb{F}_p) \curvearrowright R(\Gamma, n)$ by conjugation

$\rightsquigarrow$ take affine GIT quotient $R(\Gamma, n)_{\mathbb{F}_p} \to M_B(\Gamma, n)_{\mathbb{F}_p} := R(\Gamma, n)_{\mathbb{F}_p} /\!\!/ GL_n(\mathbb{F}_p)$

$M_B(\Gamma, n)_{\mathbb{F}_p} =: M_B$ : affine $\mathbb{F}_p$-scheme of fin type

**Properties** [DY, Prop 2.2] $k = \bar{k}$ char $k = p$. Then

- $[\rho^{ss}] = [\rho]$ , $\forall \rho : \Gamma \to GL_n(k)$
- $\forall x \in M_B$, $\exists$ semisimple rep $\rho : \Gamma \to GL_n(k)$ s.t. $x = [\rho]$.
- Given two semisimple reps $\rho : \Gamma \to GL_n(k)$ & $\rho' : \Gamma \to GL_n(k)$ s.t. $[\rho] = [\rho']$, then $\rho$ & $\rho'$ are conjugate.

There was an discussion on the representation variety $R(\Gamma, n)$ concerning the characteristic of the moduli problem it represents.

[Maret, notes on character varieties, §2.1]

Here is a concrete construction:

- For any Lie group $G$ (e.g. $GL_n(\mathbb{C})$, $GL_n(\mathbb{R})$) any f.g. group $\Gamma = \langle \gamma_1, \dots, \gamma_m \rangle$,

$$X := \{ (\phi(\gamma_1), \dots, \phi(\gamma_m)) \in G^m \mid \phi \in \mathrm{Hom}_{\mathrm{Grp}}(\Gamma, G) \} \subset \underset{\text{Lie gp top}}{G^m}$$

is homeomorphic to $\underset{\text{compact open top.}}{\mathrm{Hom}_{\mathrm{Grp}}}(\Gamma, G)$ canonically.

In particular: $X$ is independent of the choice of $\gamma_1, \dots, \gamma_m$.

- If $G$ is a $\mathbb{R}$- or $\mathbb{C}$- algebraic group, then $X$ hence $\mathrm{Hom}_{\mathrm{Grp}}(\Gamma, G)$, is an algebraic subset of $G^m$.

- In fact, this is an affine $\mathbb{Z}$-scheme of fin type. We denote it by $R(\Gamma, n)$.

# Boundedness & finiteness

**Lem** $\Gamma$ = f.g. group. $K$ nonarch loc. field.

$\rho_1, \rho_2 : \Gamma \to GL_n(\bar{K})$ two reps s.t. $\rho_1^{ss}, \rho_2^{ss}$ are conjugate.

**Then** $\rho_1$ is bounded $\Longleftrightarrow$ $\rho_2$ is bounded.

**Def**. $\rho : \Gamma \to GL_n(K)$ $K$ local field. Then

$\rho$ is called bounded, if $\rho(\Gamma)$ is contained in a maximal compact subgroup of $GL(K)$

up to conjugation, equal to $GL_n(\mathcal{O}_K)$.

**Lem** $\Gamma$ = f.g. group $k = \bar{k}$, char $p$.

**Then** $\rho(\Gamma)$ is finite $\Longleftrightarrow$ $\rho^{ss}(\Gamma)$ is finite.

How to pass from "bounded" to "finite"?

[DT, Lem 2.3 + 2.4] $M \subset M_B := M_B(\Gamma, n)_{\mathbb{F}_p}$.
constructible subset.

If $\forall$ reductive $\tau : \Gamma \to GL_n(K)$ with
$K$ local field of char $p$
and $[\tau] \in M(K)$, $\tau(\Gamma)$ is bounded
Then $\tau(\Gamma)$ is actually finite.

Moreover: $\forall \rho : \Gamma \to GL_n(L)$, $L$ any field of char $p > 0$,
$[\rho] \in M(L)$, then $\rho(\Gamma)$ is also finite
and $M$ is actually zero-dimensional.

Sketch First show $M$ zero-dimensional

In order to do this:
$\pi : R(\Gamma, n)_{\mathbb{F}_p} \longrightarrow M_B$
Take $T \subset \pi^{-1}(M)$ any irr affine defined/$\overline{\mathbb{F}_p}$
and show that $\pi(T)$ is a pt.

↙ What we will use.

<u>Cor</u> [DY, Lem 2.5]

$\varphi : \Gamma' \to \Gamma$ gp morphism of f.g. groups.

$M \subset M_B(\Gamma, n)_{\mathbb{F}_p}$ constructible.

<u>Then</u> TFAE

(1) $\forall$ reductive rep $\tau : \Gamma \to GL_n(K)$

where $K$ is local field of char $p>0$.

s.t. $[\tau] \in M(K)$, $\tau(\varphi(\Gamma'))$ is bounded

(2) $\forall$ rep $\rho : \Gamma \to GL_n(L)$

where $L$ is any field of char $p>0$.

s.t. $[\rho] \in M(L)$, $\rho(\varphi(\Gamma'))$ is finite.

# Katzarkov–Eyssidieux reduction map.

[DY, Def 1.1] **Def** $X$ normal quasi-projective variety$/\mathbb{C}$.

$\rho : \pi_1(X) \to GL_n(K)$, $K$ = non arch local field

The Katzarkov–Eyssidieux reduction map (KE map) is a holomorphic map
$$s_\rho : X \to S_\rho$$
where

- $S_\rho$ is normal & quasi-proj
- $s_\rho$ is dominant & has conn general fibers
- $\forall Z \subset X$ connected, Zar closed, TFAE
  - ⓐ $s_\rho(Z)$ is a point.
  - ⓑ $\rho(\pi_1(Z))$ is bounded
  - ⓒ $\forall$ irreducible components $Z_o \subset Z$, $\rho(\pi_1(Z_o^{norm}))$ is bounded

normalization

"UP" of KE map

Uniqueness holds only when $X$ is projective

(bc. $s_\rho$ being only dominant is not enough to rule out the possibility of further composing an open immersion)

# Katzarkov–Eyssidieux reduction map.

[DY, Def 1.1] Def $X$ normal quasi-projective variety $/\mathbb{C}$.

$M \subset M_B(\pi_1(X), n)_{\mathbb{F}_p}$ subset

The simultaneous Katzarkov–Eyssidieux reduction map (simul. KE map) is a holomorphic map

$$s_M : X \to S_M$$

where

- $S_M$ is normal & quasi-projective
- $s_M$ is dominant & has conn (gen) fibers.
- $\forall Z \subset X$ connected, Zar closed,

  $s_M(Z)$ is a point $\Longleftrightarrow$ $\forall \tau \in M(K)$, $\tau$ reductive $s_\tau(Z)$ is a point,

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$S$ normal, $X \to S$ proper, then fibers are gen conn $\Longleftrightarrow$ conn

# Construction of KE reduction map for reduced rep: Blackbox.

[CDY, Thm H] $X$ normal, quasi-projective.

$\rho: \pi_1(X) \to GL_n(K)$. reductive rep w/ $K$ nonarch. local field.

Then KE reduction map exists.

[DW, Prop 2.10] Moreover, if $X$ projective

Then 1) $s_\rho$ is proper surj, &

$S_\rho$ is normal proj

2) $\exists$ closed multivalued 1-form $\eta_\rho$

s.t. $\forall Z \subset X$ irr closed,

ⓐ $\Leftrightarrow$ ⓑ $\Leftrightarrow$ ⓒ are equivalent to

ⓓ $\eta_\rho|_{Z_{reg}}$ is trivial.

In particular: if $\rho(\pi_1(X))$ is unbounded,

then $\eta_\rho$ is nontrivial.

Last talk

**Cor.** $X$ (smooth) normal quasi-proj.

$\rho : \pi_1(X) \to GL_n(K)$ $\quad K$ = nonarch loc. field.

**Then** KE map exists.

**Sketch** Consider $\rho_{\bar{K}} : \pi_1(X) \to GL_n(\bar{K}) \rightsquigarrow \rho_{\bar{K}}^{ss}$

$\rightsquigarrow$ $\rho_{\bar{K}}^{ss}$ comes from $\rho_L^{ss} : \pi_1(X) \to GL_n(L)$

$L/K$ finite ext.

$\rightsquigarrow$ KE map of $\rho_L^{ss}$ does the job.

**Cor** $\forall\, M \subset M_B := M_B(\pi_1(X), n)_{\mathbb{F}_p}$ subset,

the simultaneous KE map of $M$ exists.

**Sketch**: simultaneous Stein factorization

always exist.

[DYK, 1.28]

## § Shafarevich morphism

**Def** $X$ normal quasi-projective variety $/\mathbb{C}$.

[DY, Thm A]
[DW, 2.12]

$\rho : \pi_1(X) \to GL_n(K)$

The Shafarevich morphism is a holomorphic map

$$sh_\rho : X \longrightarrow Sh_\rho(X)$$

where

- $Sh_\rho(X)$ is normal & q-proj
- $sh_\rho$ is dominant & conn gen fibers
- $\forall Z \subset X$ connected, Zar closed, TFAE
  - ⓐ $sh_\rho(Z)$ is a pt
  - ⓑ $\rho(\pi_1(Z))$ is finite
  - ⓒ $\forall$ irreducible components $Z_0 \subset Z$, $\rho^{ss}(\pi_1(Z_0^{norm}))$ is finite.

"UP"

nonunique when $X$ is only quasi-projective and not projective

**Rmk** if $\rho$ is large, & $sh_\rho$ exist, then $sh_\rho$ is an isom

## § Shafarevich morphism

**Def** $X$ normal quasi-projective variety $/\mathbb{C}$.

[DY, Thm 2.7]

[DW, Def 2.12(iii)]

$M \subset M_B$

The simultaneous Shafarevich morphism is a holomorphic map

$$sh_M : X \longrightarrow Sh_M(X)$$

where

- $Sh_M(X)$ is normal & q-proj
- $sh_M$ is dominant & conn gen fibers.
- $\forall Z \subset X$ connected, Zar closed, TFAE
  - ⓐ $sh_M(Z)$ is a point.
  - ⓑ $\rho(\pi_1(Z))$ is finite. $\forall \rho \in M(K)$
  - ⓒ $\forall$ irreducible components $Z_0 \subset Z$, $\rho^{ss}(\pi_1(Z_0^{norm}))$ finite. $\forall \rho \in M(K)$.

Construction of the Shafarevich morphism.

[DT, Thm 2.9] $X$ normal quasi-proj.

$\rho : \pi_1(X) \to GL_n(K)$, $K$ any field char $p$.

<u>Then</u> the Shafarevich map exists.

pf. <u>Outline</u>

<u>Step 1</u> Construct simultaneous Shafarevich map $sh_M$ for $M \subset M_B$ Zariski closed in the case $X$ smooth.

<u>Step 2</u> For a chosen $\rho$, one associate with a $M_\rho$, and $sh_{M_\rho}$ does the job. Still assuming $X$ smooth.

<u>Step 3</u> For normal $X$, reduce to smooth case by resolution of singularies.

Complements after the talk

Example (Shafarevich morphism; example by Thomas)

$X$ smooth, quasi-projective $/\mathbb{C}$.

Suppose $\rho : \pi_1(X) \to GL_n(K)$ is large, $K$ any field of char $p > 0$.

$p \in X$, $\operatorname{codim}_{\mathbb{C}}(p ; X) \geq 2 \Rightarrow d := \dim X \geq 2$

$Y = Bl_p X \longrightarrow X$

$\rightsquigarrow$ exceptional divisor $E \simeq \mathbb{P}^{d-1}$

$\Rightarrow \pi_1(E) = 0 \Rightarrow \rho(\pi_1(E)) = 0$

Hence the Shafarevich map $sh_\rho : Y \to Sh_\rho(Y)$ of $Y$ factors through $X$.

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & Sh_\rho(Y) \\ & X \arrow[u] \end{tikzcd}$$

On the other hand, $\rho$ large $\Rightarrow sh_X$ is iso.

$\Rightarrow Y \to X$ is the Shafarevich map of $Y$.